

# Universal estimates for parabolic equations and applications for non-linear and non-local problems

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## Abstract

We obtain some "universal" estimates for  $L_2$ -norm of the solution of a parabolic equation via a weighted version of  $H^{-1}$ -norm of the free term. More precisely, we found the limit upper estimate that can be achieved by transformation of the equation by adding a constant to the zero order coefficient. The inverse matrix of the higher order coefficients of the parabolic equation is included into the weight for the  $H^{-1}$ -norm. The constant in the estimate obtained is independent from the choice of the dimension, domain, and the coefficients of the parabolic equation, it is why it can be called an universal estimate. As an example of applications, we found an asymptotic upper estimate for the norm of the solution at initial time. As an another example, we established existence and regularity for non-linear and non-local problems.

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## 1 Introduction

We study prior estimates for first boundary value problems for parabolic equations. The classical results for these equation give upper estimate for the  $L_2$ -type Sobolev norm of the solution via a  $H^{-1}$ -norm of the nonhomogeniuos term, where  $H^{-1}$  is the space being dual to the space  $\overset{0}{W}_2^1(D)$  (see, e.g., the first energy inequality in Ladyzhenskaia (1985)). We suggest a modification of this estimate.

We found the limit minimal upper estimate that can be achieved by varying the zero order coefficient of the original equation by adding a constant. In other words, we study the case when the original equation is transformed into a new one such that the original solution  $u(x, t)$  is to be replaced by  $u(x, t)e^{-Kt}$ ; the value of  $K$  is being varied (Theorem 3.1 and Lemma 8.1). The constant in the estimate is the same for all possible choices of the dimension, domain, time horizon, and the coefficients of the parabolic equations. It is why it can be called a universal estimate. These results represent an important development of the extension of the results from Dokuchaev (2008), where an "universal" estimate was obtained for the gradient via  $L_2$ -norm of the nonhomogeneous term. In contrast, the present paper gives the estimate of the  $L_2$ -norm via a  $H^{-1}$ -type norm of the nonhomogeneous term, i.e., via a weaker norm. It is shown that the estimate obtained is sharp (Theorem 7.1).

As an example of applications, we obtained a sharp asymptotic upper estimate for the solution at initial time (Theorems 5.2 and 7.2). The constant in this estimate is again the same for all possible equations. As another example of applications, we suggest a new approach for establishing of existence and regularity for non-linear and non-local parabolic equations (Theorem 6.1). We found an explicit sufficient conditions for existence and regularity (Conditions (6.3) or (6.4)). These conditions are easy to verify, and they cover a wide class of non-linear and non-local parabolic equations.

## 2 Definitions

### Spaces and classes of functions.

We denote by  $|\cdot|$  the Euclidean norm in  $\mathbf{R}^k$  and the Frobenius norm in  $\mathbf{R}^{k \times m}$ , and we denote by  $\bar{G}$  denote the closure of a region  $G \subset \mathbf{R}^k$ .

We denote by  $\|\cdot\|_X$  the norm in a linear normed space  $X$ , and  $(\cdot, \cdot)_X$  denote the scalar product in a Hilbert space  $X$ . For a Banach space  $X$ , we denote by  $C([a, b], X)$  the Banach space of continuous functions  $x : [a, b] \rightarrow X$ .

Let  $G \subset \mathbf{R}^k$  be an open domain, then  $W_q^m(G)$  denote the Sobolev space of functions that belong  $L_q(G)$  together with the distributional derivatives up to the  $m$ th order,  $q \geq 1$ .

We are given an open domain  $D \subseteq \mathbf{R}^n$  such that either  $D = \mathbf{R}^n$  or  $D$  is bounded with  $C^2$ -smooth boundary  $\partial D$ .

Let  $T > 0$  be given, and let  $Q \triangleq D \times (0, T)$ .

Let  $H^0 \triangleq L_2(D)$ , and let  $H^1 \triangleq W_2^1(D)$  be the closure in the  $W_2^1(D)$ -norm of the set of all

smooth functions  $u : D \rightarrow \mathbf{R}$  such that  $u|_{\partial D} \equiv 0$ . The spaces  $H^k$  are Hilbert spaces, and  $H^1$  is a closed subspace of  $W_2^1(D)$ . Let  $H^2 = W_2^2(D) \cap H^1$  be the space equipped with the norm of  $W_2^2(D)$ . The spaces  $H^k$  are Hilbert spaces, and  $H^k$  is a closed subspace of  $W_2^k(D)$ ,  $k = 1, 2$ .

Let  $H^{-1}$  be the dual space to  $H^1$ , with the norm  $\|\cdot\|_{H^{-1}}$  such that if  $u \in H^0$  then  $\|u\|_{H^{-1}}$  is the supremum of  $(u, w)_{H^0}$  over all  $w \in H^1$  such that  $\|w\|_{H^1} \leq 1$ .  $H^{-1}$  is a Hilbert space.

We will write  $(u, w)_{H^0}$  for  $u \in H^{-1}$  and  $w \in H^1$ , meaning the obvious extension of the bilinear form from  $u \in H^0$  and  $w \in H^1$ .

We denote by  $\bar{\ell}_1$  the Lebesgue measure in  $\mathbf{R}$ , and we denote by  $\bar{\mathcal{B}}_1$  the  $\sigma$ -algebra of Lebesgue sets in  $\mathbf{R}^1$ .

For  $k = -1, 0, 1, 2$ , we introduce the spaces

$$X^k \triangleq L^2([0, T], \bar{\mathcal{B}}_1, \bar{\ell}_1; H^k), \quad \mathcal{C}^k \triangleq C([0, T]; H^k).$$

We introduce the spaces

$$Y^k \triangleq X^k \cap \mathcal{C}^{k-1}, \quad k = 0, 1, 2,$$

with the norm  $\|u\|_{Y^k} \triangleq \|u\|_{X^k} + \|u\|_{\mathcal{C}^{k-1}}$ .

We use the notations

$$\nabla u \triangleq \left( \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n} \right)^\top, \quad \nabla \cdot U = \sum_{i=1}^n \frac{\partial U_i}{\partial x_i}$$

for functions  $u : \mathbf{R}^n \rightarrow \mathbf{R}$  and  $U = (U_1, \dots, U_n)^\top : \mathbf{R}^n \rightarrow \mathbf{R}^n$ . In addition, we use the notation

$$(U, V)_{H^0} = \sum_{i=1}^n (U_i, V_i)_{H^0}, \quad \|U\|_{H^0} = (U, U)_{H^0}^{1/2}$$

for functions  $U, V : D \rightarrow \mathbf{R}^n$ , where  $U = (U_1, \dots, U_n)$  and  $V = (V_1, \dots, V_n)$ .

## The boundary value problem

We consider the following problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mathcal{A}u + \varphi, & t \in (0, T), \\ u(x, 0) &= 0, & u(x, t)|_{x \in \partial D} = 0. \end{aligned} \tag{2.1}$$

Here  $u = u(x, t)$ ,  $(x, t) \in Q$ , and

$$\mathcal{A}y \triangleq \sum_{i=1}^n \frac{\partial}{\partial x_i} \sum_{j=1}^n \left( b_{ij}(x, t) \frac{\partial y}{\partial x_j}(x) \right) + \sum_{i=1}^n f_i(x, t) \frac{\partial y}{\partial x_i}(x) + \lambda(x, t)y(x), \tag{2.2}$$

where  $b(x, t) : \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}^{n \times n}$ ,  $f(x, t) : \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}^n$ , and  $\lambda(x, t) : \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}$ , are bounded measurable functions, and  $b_{ij}, f_i, x_i$  are the components of  $b, f$ , and  $x$ . The matrix  $b = b^\top$  is symmetric.

To proceed further, we assume that Conditions 2.1-2.2 remain in force throughout this paper.

**Condition 2.1** *There exists a constant  $\delta > 0$  such that*

$$\xi^\top b(x, t) \xi \geq \delta |\xi|^2 \quad \forall \xi \in \mathbf{R}^n, (x, t) \in Q. \quad (2.3)$$

Inequality (2.3) means that equation (2.1) is coercive.

**Condition 2.2** *The functions  $b(x, t) : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^{n \times n}$ ,  $f(x, t) : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n$ ,  $\lambda(x, t) : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}$ , are measurable, and*

$$\operatorname{ess\,sup}_{(x,t) \in Q} \left[ |b(x, t)| + |b(x, t)^{-1}| + |f(x, t)| + |\lambda(x, t)| \right] < +\infty.$$

We introduce the sets of parameters

$$\begin{aligned} \mu &\triangleq (T, n, D, \delta, v, f, \lambda), \\ \mathcal{P} = \mathcal{P}(\mu) &\triangleq \left( T, n, D, \delta, \operatorname{ess\,sup}_{(x,t) \in Q} \left[ |b(x, t)| + |f(x, t)| + |\lambda(x, t)| \right] \right). \end{aligned}$$

We consider all possible  $\mu$  such that the conditions imposed above are satisfied.

### 3 Special estimates for the solution

We assume that  $\varphi \in X^{-1}$ . This means that there exist functions  $F = (F_1, \dots, F_n) : Q \rightarrow \mathbf{R}^n$  and  $F_0 : Q \rightarrow \mathbf{R}$  such that  $F_k \in X^0 = L_2(Q)$ ,  $k = 0, 1, \dots, n$ , and

$$\varphi(x, t) = \nabla \cdot F(x, t) + F_0(x, t). \quad (3.1)$$

In other words,  $\varphi(x, t) = \sum_{k=1}^n \frac{\partial F_k}{\partial x_k}(x, t) + F_0(x, t)$ .

The classical solvability results for the parabolic equations give that there exists a unique solution  $u \in Y^1$  of problem (2.1) for any  $\varphi \in X^{-1}$ . In addition, it follows from the first energy inequality (or the first fundamental inequality) that, for any  $K \in \mathbf{R}$  and  $M \geq 0$ , there exist constants  $\tilde{C}_i(K, M, \mathcal{P}) > 0$ ,  $i = 0, 1$ , such that

$$e^{-2Kt} \|u(\cdot, t)\|_{H^0}^2 + M \int_0^t e^{-2Ks} \|u(\cdot, s)\|_{H^0}^2 ds$$

$$\begin{aligned} &\leq \tilde{C}_1(K, M, \mathcal{P}) \int_0^t e^{-2Ks} (F(\cdot, s), b(\cdot, s)^{-1} F(\cdot, s))_{H^0} ds \\ &\quad + \tilde{C}_0(K, M, \mathcal{P}) \int_0^t e^{-2Ks} \|F_0(\cdot, s)\|_{H^0}^2 ds \quad \forall \varphi \in X^{-1}, t \in (0, T], \end{aligned} \quad (3.2)$$

where  $F_i \in X^0$  are such that (3.1) holds. (See, e.g., estimate (3.14) from Ladyzhenskaia (1985), Chapter III, §3). We have used here the following obvious estimate

$$\sum_{k=1}^n \|F_k(\cdot, s)\|_{H^0}^2 ds \leq c(F(\cdot, s), b(\cdot, s)^{-1} F(\cdot, s))_{H^0},$$

where  $c = c(\mathcal{P}) > 0$  is a constant.

Let  $C_i(K, M, \mathcal{P}) \triangleq \inf \tilde{C}_i(K, M, \mathcal{P})$ , where the infimum is taken over all  $\tilde{C}_i(K, M, \mathcal{P})$  such that (3.2) holds.

### Theorem 3.1

$$\sup_{\mu, M \geq 0} \inf_{K \geq 0} C_1(K, M, \mathcal{P}(\mu)) \leq \frac{1}{2}, \quad \sup_{\mu, M \geq 0} \inf_{K \geq 0} C_0(K, M, \mathcal{P}(\mu)) = 0.$$

**Corollary 3.1** *For any  $\mu$  and any  $M > 0$ ,  $\varepsilon > 0$ , there exists  $K = K(\varepsilon, M, \mathcal{P}(\mu)) \geq 0$  such that*

$$\begin{aligned} &\sup_{s \in [0, t]} e^{-2Ks} \|u(\cdot, s)\|_{H^0}^2 + M \int_0^t e^{-2Ks} \|u(\cdot, s)\|_{H^0}^2 ds \\ &\leq \left(\frac{1}{2} + \varepsilon\right) \sum_{k=0}^n \int_0^t (F(\cdot, s), b(\cdot, s)^{-1} F(\cdot, s))_{H^0} ds + \varepsilon \int_0^t \|F_0(\cdot, s)\|_{H^0}^2 ds \quad (3.3) \\ &\quad \forall t \in [0, T], \varphi \in X^{-1}, \end{aligned}$$

where  $u$  is the solution of problem (2.1), and where  $F_i \in X^0$  are such that (3.1) holds.

## 4 The case of non-linear and non-local equations

Let us consider the following mapping  $\mathcal{N}(v) : Y^1 \rightarrow X^{-1}$  such that

$$\begin{aligned} \mathcal{N}(v) \triangleq & \sum_{i=1}^n \frac{\partial}{\partial x_i} \sum_{j=1}^n \left( \widehat{b}_{ij}(v(\cdot), x, t) \frac{\partial v}{\partial x_j}(x, t) \right) + \sum_{i=1}^n \widehat{f}_i(v(\cdot), x, t) \frac{\partial v}{\partial x_i}(x, t) \\ & + \widehat{\lambda}(v(\cdot), x, t) v(x, t) + \widehat{\varphi}(v(\cdot), x, t), \end{aligned} \quad (4.1)$$

where  $\widehat{b}(v(\cdot), x, t) : Y^1 \times Q \rightarrow \mathbf{R}^{n \times n}$ ,  $\widehat{f}(v(\cdot), x, t) : Y^1 \times Q \rightarrow \mathbf{R}^n$ ,  $\widehat{\lambda}(v(\cdot), x, t) : Y^1 \times Q \rightarrow \mathbf{R}$ , are bounded functions, and  $b_{ij}, f_i, x_i$  are the components of  $b, f$ , and  $x$ . The function  $\widehat{\varphi}(v(\cdot), x, t)$  is defined on  $Y^1 \times Q \rightarrow \mathbf{R}$  and belongs to  $X^{-1}$  for any given  $v(\cdot) \in Y^1$ . The matrix  $\widehat{b} = \widehat{b}^\top$  is symmetric.

**Corollary 4.1** *Let  $u \in Y^1$  be a solution of the problem*

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mathcal{N}(u), & t \in (0, T), \\ u(x, 0) &= 0, & u(x, t)|_{x \in \partial D} = 0. \end{aligned} \quad (4.2)$$

*such that Conditions 2.1-2.2 are satisfied for  $b(x, t) = \widehat{b}(u(\cdot), x, t)$ ,  $f(x, t) = \widehat{f}(u(\cdot), x, t)$ , and  $\lambda(x, t) = \widehat{\lambda}(u(\cdot), x, t)$ , and such that  $\varphi(x, t) \triangleq \widehat{\varphi}(u(\cdot), x, t)$  belongs to  $X^{-1}$  and is such that (3.1) holds for  $F_i \in X^0$ . Then, for any  $M > 0$  and  $\varepsilon > 0$ , there exists  $K = K(\varepsilon, M, \mathcal{P}) \geq 0$  such that (3.3) holds.*

Note that the parabolic equation in (4.2) is non-linear and non-local.

Corollary 4.1 does not establish existence. Some existence results for non-local and non-linear problems are given below.

## 5 Applications: asymptotic estimate at initial time

Let

$$X_c^0 \triangleq \left\{ \varphi \in X^0 : \lim_{t \rightarrow 0+} \frac{1}{t} \int_0^t \|\varphi(\cdot, s)\|_{H^0}^2 ds = \|\varphi(\cdot, 0)\|_{H^0}^2 \right\}. \quad (5.1)$$

Note that the condition that  $\varphi \in X_c^0$  is not restrictive for  $\varphi \in X^0$ ; for instance, it holds if  $s = 0$  is a Lebesgue point for  $\|\varphi(\cdot, s)\|_{H^0}^2$ .

**Theorem 5.1** *Let  $\varphi \in X_c^0$ . Then, for any admissible  $\mu$ , the solution  $u$  of problem (2.1) is such that*

$$\overline{\lim}_{t \rightarrow 0+} \sup_{\varphi \in X_c^{-1}} \frac{1}{t} \frac{\|u(\cdot, t)\|_{H^0}^2}{\|\varphi(\cdot, 0)\|_{H^0}^2} = 0, \quad (5.2)$$

where  $u$  is the solution of problem (2.1) for the corresponding  $\varphi$ .

Further, let

$$\begin{aligned} X_c^{-1} &\triangleq \left\{ \varphi \in X^{-1} : \text{there exists a set } \{F_k\}_{k=1}^n \subset X^0 \text{ such that (3.1) holds with } F_0 \equiv 0, \right. \\ &\quad \left. \text{and } \lim_{t \rightarrow 0+} \frac{1}{t} \int_0^t (F(\cdot, s), b(\cdot, s)^{-1} F(\cdot, s))_{H^0} ds = (F(\cdot, 0), b(\cdot, 0)^{-1} F(\cdot, 0))_{H^0} \right\}. \end{aligned} \quad (5.3)$$

Here  $F = (F_1, \dots, F_n)$ . Again, the limit condition in (5.3) is not restrictive; for instance, it holds if  $s = 0$  is a Lebesgue point for  $(F(\cdot, s), b(\cdot, s)^{-1} F(\cdot, s))_{H^0}$ . Clearly, the set  $X_c^{-1}$  is non-empty if some mild conditions of regularity in  $t$  are satisfied for  $b(x, t)^{-1}$ .

**Theorem 5.2** *Let  $\varphi \in X_c^{-1}$  and let  $F_k \in X^0$ ,  $k = 0, 1, \dots, n$ , be the corresponding functions presented in (5.3) with  $F_0 = 0$ . Then, for any admissible  $\mu$ ,*

$$\overline{\lim}_{t \rightarrow 0+} \sup_{\varphi \in X_c^{-1}} \frac{1}{t} \frac{\|u(\cdot, t)\|_{H^0}^2}{(F(\cdot, 0), b(\cdot, 0)^{-1} F(\cdot, 0))_{H^0}} \leq \frac{1}{2}, \quad (5.4)$$

where  $u$  is the solution of problem (2.1) for the corresponding  $\varphi$ ,  $F = (F_1, \dots, F_n)$ .

**Corollary 5.1** *Let  $u \in Y^1$  be a solution of problem (4.2) such that the assumptions of Corollary 4.1 are satisfied. Let  $\varphi(x, t) = \widehat{\varphi}(u(\cdot), x, t)$  be such that  $\varphi = F_0 + \tilde{\varphi}$ , where  $F_0 \in X_c^0$  and  $\tilde{\varphi} \in X_c^{-1}$ , and let  $F_i \in X^0$  be the corresponding functions presented in (3.1) such that the limit conditions from (5.3) are satisfied. Then*

$$\overline{\lim}_{t \rightarrow 0+} \frac{1}{t} \|u(\cdot, t)\|_{H^0}^2 \leq \frac{1}{2} (F(\cdot, 0), b(\cdot, 0)^{-1} F(\cdot, 0))_{H^0}, \quad (5.5)$$

where  $F = (F_1, \dots, F_n)$ .

Note that  $F_0$  is not being presented in the last estimate.

## 6 Applications: existence for non-linear and non-local equations

The universal estimates from Theorem 3.1 can be also applied to analysis of non-linear and non-local parabolic equations. These equations have many applications, and they were intensively studied (see. e.g., Ammann, (2005), Ladyzenskaya *et al* (1967), Zheng (2004), and references there). Theorem 3.1 gives a new way to establish conditions of solvability of these equations. This approach covers many cases when the solutions and the gradient are included into the non-local and non-linear term.

Let  $B(u(\cdot)) : X^0 \rightarrow X^{-1}$  be a mapping that describes non-linear and non-local term in the equation.

Let us consider the following boundary value problem in  $Q$ :

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mathcal{A}u + B(u) + \varphi, & t \in (0, T), \\ u(x, 0) &= 0, & u(x, t)|_{x \in \partial D} = 0. \end{aligned} \quad (6.1)$$

Here  $\mathcal{A}$  is the linear operator defined above. For  $K > 0$ , introduce the mappings

$$B_K(u) \triangleq e^{-Kt} B(\bar{u}_K), \quad \text{where} \quad \bar{u}_K(x, t) \triangleq e^{Kt} u(x, t). \quad (6.2)$$

**Theorem 6.1** Assume that  $B(u)$  maps  $X^0$  into  $X^{-1}$ . Moreover, assume that there exist constants  $K_* > 0$  and  $C_* > 0$  such that

$$\|B_K(u_1) - B_K(u_2)\|_{X^{-1}} \leq C_* \|u_1 - u_2\|_{X^0} \quad \forall u_1, u_2 \in X^0, \quad K > K_*. \quad (6.3)$$

Then there exists a unique solution  $u \in Y^1$  of problem (6.1) for any  $\varphi \in X^{-1}$ .

**Theorem 6.2** Assume that  $B(u)$  maps  $X^1$  into  $X^0$  and that there exist constants  $K_* > 0$  and  $C_* > 0$  such that

$$\|B_K(u_1) - B_K(u_2)\|_{X^0} \leq C_* \|u_1 - u_2\|_{X^1} \quad \forall u_1, u_2 \in X^1, \quad K > K_*. \quad (6.4)$$

Then there exists a unique solution  $u \in Y^2$  of problem (6.1) for any  $\varphi \in X^0$ .

### Examples of admissible $B$

Some examples covered by Theorem 6.1 are listed below.

**Theorem 6.3** The assumptions of Theorem 6.1 hold for the following mappings  $B(u)$ :

(i) A local non-linearity:

$$B(u) = \beta(u(x, t), x, t), \quad (6.5)$$

where  $\beta : \mathbf{R} \times Q \rightarrow \mathbf{R}$  is a measurable function such that  $\beta(0, \cdot) \in L_2(Q)$  and that there exists a constant  $C_L > 0$  such that

$$|\beta(z_1, x, t) - \beta(z_2, x, t)| \leq C_L |z_1 - z_2| \quad \forall z_1, z_2 \in \mathbf{R}, \quad x, t. \quad (6.6)$$

(ii) A distributional non-linearity:

$$B(u) \triangleq \nabla \cdot \beta(u(x, t), x, t), \quad (6.7)$$

where  $\beta : \mathbf{R} \times Q \rightarrow \mathbf{R}^n$  is a measurable function such that  $\beta(0, \cdot) \in L_2(Q)$  and (6.6) holds.

(iii) A non-local non-linearity (integral nonlinearity):

$$(B(u))(x, t) = \int_D \beta(u(y, t), x, t, y) dy,$$

where  $\beta : \mathbf{R} \times Q \times D \rightarrow \mathbf{R}$  is a measurable function such that  $\int_D \beta(0, x, t, y) dy \in L_2(Q)$  as a function of  $(x, t)$ , and there exists a constant  $C_L > 0$  such that

$$|\beta(z_1, x, t, y) - \beta(z_2, x, t, y)| \leq C_L |z_1 - z_2| \quad \forall z_1, z_2 \in \mathbf{R}, \quad x, t, y. \quad (6.8)$$

We assume here that  $D$  is a bounded domain.



(iv) A non-local in space distributional non-linearity:

$$(B(u))(x, t) = \nabla \cdot \int_D \beta(u(y, t), x, t, y) dy,$$

where  $\beta : \mathbf{R} \times Q \times D \rightarrow \mathbf{R}^n$  is a measurable function such that  $\int_D \beta(0, \cdot, y) dy \in L_2(Q)$  as a function of  $(x, t)$ , and (6.8) holds. We assume here that  $D$  is a bounded domain.

(v) A non-local in time and space non-linearity:

$$(B(u))(x, t) = \int_0^t ds \int_D \beta(u(y, s), x, t, y, s) dy,$$

where  $\beta : \mathbf{R} \times Q^2 \rightarrow \mathbf{R}$  is a measurable function such that  $\int_0^t ds \int_D \beta(0, x, t, y, s) dy \in L_2(Q)$  as a function of  $(x, t)$ , and there exists a constant  $C_L > 0$  such that

$$|\beta(z_1, x, t, y, s) - \beta(z_2, x, t, y, s)| \leq C_L |z_1 - z_2| \quad \forall z_1, z_2 \in \mathbf{R}, \quad x, t, y, s. \quad (6.9)$$

We assume here that  $D$  is a bounded domain.

(vi) A non-local in time and space distributional non-linearity:

$$(B(u))(x, t) = \nabla \cdot \int_0^t ds \int_D \beta(u(y, s), x, t, y, s) dy,$$

where  $\beta : \mathbf{R} \times Q^2 \rightarrow \mathbf{R}^n$  is a measurable function such that  $\int_0^t ds \int_D \beta(0, \cdot, y, s) dy \in L_2(Q)$  as a function of  $(x, t)$ , and (6.9) holds. We assume here that  $D$  is a bounded domain.

(vii) Nonlinear delay parabolic equations:

$$(B(u))(x, t) \triangleq \nabla \cdot \beta(u(x, \tau(t)), x, \tau(t)) + \bar{\beta}(u(x, \tau(t)), x, \tau(t)). \quad (6.10)$$

Here  $\tau(\cdot) : [0, T] \rightarrow \mathbf{R}$  is a given measurable function such that  $\tau(t) \in [0, t]$ , and that there exists  $\theta \in [0, T]$  such that  $\tau(t) = 0$  for  $t < \theta$ , the function  $\tau(\cdot) : [\theta, T] \rightarrow \mathbf{R}$  is non-decreasing and absolutely continuous, and  $\text{ess sup}_{t \in [\theta, T]} \left| \frac{d\tau}{dt}(t) \right|^{-1} < +\infty$ . The functions  $\beta : \mathbf{R} \times \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}^n$  and  $\bar{\beta} : \mathbf{R} \times \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}$  are bounded and measurable. In addition, we assume that the derivative  $\frac{\partial \beta}{\partial x}(x, t)$  is bounded,  $\beta(0, \cdot) \in L_2(Q)$ ,  $\hat{\beta}(0, \cdot) \in L_2(Q)$ , and there exists a constant  $C_L > 0$  such that

$$|\beta(z_1, x, t) - \beta(z_2, x, t)| + |\hat{\beta}(z_1, x, t) - \hat{\beta}(z_2, x, t)| \leq C_L |z_1 - z_2| \quad \forall z_1, z_2 \in \mathbf{R}, \quad x, t. \quad (6.11)$$

(viii) Non-local term for the backward Kolmogorov equations for a jump diffusion process:

$$(Bu)(x, t) \triangleq \int_{\mathbf{R}^n} \mathbb{I}_{\{x+c(x, y, t) \in D\}} (u(x + c(x, y, t), t) - u(x, t) - c(x, y, t)^\top \nabla u(x, t)) \rho(y, t) dy.$$

Here  $\rho(y, t) : \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}$  is a function such that  $\rho(\cdot) \in L_\infty([0, T], \ell_1, \bar{\mathcal{B}}_1, L_1(\mathbf{R}^n))$ . The function  $c(x, y, t) : D \times \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}^n$  is measurable, bounded, and such that the derivative  $\frac{\partial c}{\partial x}(x, y, t)$  is bounded, the derivative  $\frac{\partial c}{\partial z}(x, y, t)$  exists almost everywhere, and there exists a uniquely defined function  $\psi : D \times \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}^n$  such that  $z = x + c(x, y, t)$  for  $y = \psi(x, z, t)$ . In addition, we assume that  $\text{ess sup}_{t \in [0, T]} \int_{D \times D} |r(x, z, t)|^2 dx dz < +\infty$ , where  $r(x, z, t) \triangleq \rho(\psi(x, z, t), t) \frac{\partial \psi}{\partial z}(x, z, t)$ .

Clearly, linear combinations of the non-linear and non-local terms listed above are also covered, as well as terms formed as compound mappings.

## 7 On the sharpness of the estimates

**Theorem 7.1** *There exists a set of parameters  $(n, D, b(\cdot), f(\cdot), \lambda(\cdot))$  such that, for any  $T > 0$ ,  $M \geq 0$ ,*

$$\inf_{K \geq 0} C(K, M, \mathcal{P}(\mu)) = \frac{1}{2}. \quad (7.1)$$

for  $\mu = (T, n, D, b(\cdot), f(\cdot), \lambda(\cdot))$ .

**Theorem 7.2** *There exists a set of parameters  $(n, D, b(\cdot), f(\cdot), \lambda(\cdot))$  such that*

$$\overline{\lim}_{t \rightarrow 0+} \sup_{\varphi \in X_c^{-1}} \frac{1}{t} \frac{\|u(\cdot, t)\|_{H^0}^2}{(F(\cdot, 0), b(\cdot, 0)^{-1} F(\cdot, 0))_{H^0}} = \frac{1}{2}. \quad (7.2)$$

where  $u$  is the solution of problem (2.1) for the corresponding  $\varphi \in X_c^{-1}$ , and where  $F = (F_1, \dots, F_n)$  with  $F_i \in X^0$  being the corresponding functions presented in (5.3).

## 8 Proofs

**Lemma 8.1** *For any admissible  $\mu$  and any  $\varepsilon > 0$ ,  $M > 0$ , there exists  $\tilde{K} = \tilde{K}(\varepsilon, M, \mathcal{P}(\mu)) \geq 0$  such that*

$$\begin{aligned} & \|u(\cdot, t)\|_{H^0}^2 + M \int_0^t \|u(\cdot, s)\|_{H^0}^2 ds \\ & \leq \left(\frac{1}{2} + \varepsilon\right) \int_0^t \left((F(\cdot, s), b(\cdot, s)^{-1} F(\cdot, s))_{H^0} ds + \varepsilon \int_0^t \|F_0(\cdot, s)\|_{H^0}^2 ds \right) \end{aligned} \quad (8.1)$$

for all  $K \geq \tilde{K}(\varepsilon, M, \mathcal{P})$ ,  $t \in (0, T]$ , for all  $\varphi \in X^{-1}$  represented as (3.1) with  $F_i \in X^0$ . Here  $u \in Y^1$  is the solution of the boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mathcal{A}u - Ku + \varphi, \quad t \in (0, T), \\ u(x, 0) &= 0, \quad u(x, t)|_{x \in \partial D} = 0. \end{aligned} \quad (8.2)$$

Uniqueness and existence of solution  $u \in Y^1$  of problem (8.2) follows from the classical results (see, e.g., Ladyzhenskaia (1985), Chapter III).

*Proof of Lemma 8.1.* Clearly,  $\mathcal{A}u = \mathcal{A}_s u + \mathcal{A}_r u$ , where

$$\mathcal{A}_s u = \nabla \cdot (b \nabla u) = \sum_{i=1}^n \frac{\partial}{\partial x_i} \sum_{j=1}^n \left( b_{ij} \frac{\partial u}{\partial x_j} \right), \quad \mathcal{A}_r u = \sum_{i=1}^n f_i \frac{\partial u}{\partial x_i} + \lambda u.$$

Assume that  $\varphi(\cdot, t)$  is differentiable and has a compact support inside  $D$  for all  $t$ . We have that

$$\begin{aligned} \|u(\cdot, t)\|_{H^0}^2 - \|u(\cdot, 0)\|_{H^0}^2 &= (u(\cdot, t), u(\cdot, t))_{H^0} - (u(\cdot, 0), u(\cdot, 0))_{H^0} \\ &= 2 \int_0^t \left( u, \frac{\partial u}{\partial s} \right)_{H^0} ds = 2 \int_0^t (u, \mathcal{A}u - Ku + \varphi)_{H^0} ds \\ &= 2 \int_0^t (u, \nabla \cdot (b \nabla u))_{H^0} ds + 2 \int_0^t (u, \mathcal{A}_r u)_{H^0} ds - 2K \int_0^t (u, u)_{H^0} ds \\ &\quad + 2 \int_0^t (u, \varphi)_{H^0} ds. \end{aligned} \quad (8.3)$$

Let arbitrary  $\varepsilon_0 > 0$  and  $\widehat{\varepsilon}_0 > 0$  be given. Let  $v \triangleq \sqrt{b}$ , i.e.,  $b = v^2$ ,  $v = v^\top$ . We have that

$$\begin{aligned} 2(u, \varphi)_{H^0} &= 2(u, \nabla \cdot F)_{H^0} + 2(u, F_0)_{H^0} = -2(v \nabla u, v^{-1} F)_{H^0} + 2(u, F_0)_{H^0} \\ &\leq \frac{2}{1 + 2\varepsilon_0} (v \nabla u, v \nabla u)_{H^0}^2 + \left( \frac{1}{2} + \varepsilon_0 \right) \|v^{-1} F\|_{H^0}^2 + \frac{1}{\widehat{\varepsilon}_0} \|u\|_{H^0}^2 + \widehat{\varepsilon}_0 \|F_0\|_{H^0}^2 \\ &= \frac{2}{1 + 2\varepsilon_0} (\nabla u, b \nabla u)_{H^0}^2 + \left( \frac{1}{2} + \varepsilon_0 \right) (F, b^{-1} F)_{H^0} + \frac{1}{\widehat{\varepsilon}_0} \|u\|_{H^0}^2 + \widehat{\varepsilon}_0 \|F_0\|_{H^0}^2, \end{aligned} \quad (8.4)$$

and

$$2(u, \nabla \cdot (b \nabla u))_{H^0} = -2(\nabla u, b \nabla u)_{H^0}. \quad (8.5)$$

In addition, we have that in under the integrals in (8.3),

$$2(u, \mathcal{A}_r u)_{H^0} \leq \varepsilon_1^{-1} \|u\|_{H^0}^2 + \varepsilon_1 \|\mathcal{A}_r u\|_{H^0}^2 \quad \forall \varepsilon_1 > 0.$$

By the first energy inequality, there exist constants  $c'_* = c'_*(\mathcal{P}) > 0$  and  $c_* = c_*(\mathcal{P}) > 0$  such that

$$\int_0^t \|u(\cdot, s)\|_{H^1}^2 ds \leq c'_* \sum_{k=0}^n \int_0^t \|F_k(\cdot, s)\|_{H^0}^2 ds \leq c_* \int_0^t (F, b^{-1} F)_{H^0} ds. \quad (8.6)$$

(See, e.g. inequality (3.14) from Ladyzhenskaia (1985), Chapter III). Moreover, this constant  $c_*$  can be taken the same for all  $t \in [0, T]$  and all  $K > 0$ . Further, there exists a constant  $c_1 = c_1(\mathcal{P}) > 0$  such that

$$2(u, \mathcal{A}_r u)_{H^0} \leq \varepsilon_1^{-1} \|u\|_{H^0}^2 + c_1 \varepsilon_1 \|u\|_{H^1}^2.$$

It follows that

$$2 \int_0^t (u, \mathcal{A}_r u)_{H^0} ds \leq \varepsilon_1^{-1} \int_0^t \|u\|_{H^0}^2 ds + \varepsilon_0 \int_0^t (F, b^{-1} F)_{H^0} ds, \quad (8.7)$$

if  $\varepsilon_1 > 0$  is taken such that  $c_1 c_* \varepsilon_1 = \varepsilon_0$ .

By (8.3)-(8.7), it follows that

$$\begin{aligned} & \|u(\cdot, t)\|_{H^0}^2 + M \int_0^t \|u(\cdot, s)\|_{H^0}^2 ds \\ & \leq \left[ \frac{2}{1+2\varepsilon_0} - 2 \right] \int_0^t (\nabla u, b \nabla u)_{H^0} ds + [\varepsilon_1^{-1} + \widehat{\varepsilon}_0^{-1} + M - 2K] \int_0^t \|u\|_{H^0}^2 ds \\ & + \left( \frac{1}{2} + 2\varepsilon_0 \right) \int_0^t (F, b^{-1} F)_{H^0} ds + (\varepsilon_0 + \widehat{\varepsilon}_0) \int_0^t \|F_0(\cdot, s)\|_{H^0}^2 ds \\ & \leq \left( \frac{1}{2} + 2\varepsilon_0 \right) \int_0^t (F, b^{-1} F)_{H^0} ds + (\varepsilon_0 + \widehat{\varepsilon}_0) \int_0^t \|F_0(\cdot, s)\|_{H^0}^2 ds, \end{aligned} \quad (8.8)$$

if  $2K > \varepsilon_1^{-1} + c'_v + M$ . Then the proof of Lemma 8.1 follows.  $\square$

*Proof of Theorem 3.1.* Clearly,  $u(x, t) = e^{Kt} u_K(x, t)$ , where  $u$  is the solution of problem (2.1) and  $u_K$  is the solution of (8.2) for the nonhomogeneous term  $e^{-Kt} \varphi(x, t)$ . Therefore, Theorem 3.1 follows immediately from Lemma 8.1.  $\square$

Corollary 3.1 follows immediately from Theorem 3.1.

Corollary 4.1 follows immediately from Corollary 3.1.

*Proof of Theorem 5.1.* Let  $\varepsilon > 0$  be given. By Corollary 3.1, there exists  $K(\varepsilon) = K(\varepsilon, \mathcal{P}(\mu))$  such that

$$e^{-2K(\varepsilon)t} \|u(\cdot, t)\|_{H^0}^2 \leq \varepsilon \int_0^t e^{-2K(\varepsilon)s} \|\varphi(\cdot, s)\|_{H^0}^2 ds \quad \forall t \in (0, T). \quad (8.9)$$

Let  $\varphi \in X_c^{-1}$ . Set

$$p_0(\varphi, t) \triangleq \frac{1}{t} \int_0^t \|\varphi(\cdot, s)\|_{H^0}^2 ds, \quad q(u, t) \triangleq \|u(\cdot, t)\|_{H^0}.$$

It follows that

$$\sup_h \left( \frac{q(u, t)}{tp_0(\varphi, t)} - \frac{1 - e^{-2K(\varepsilon)t}}{tp_0(\varphi, t)} q(u, t) \right) \leq \varepsilon \quad \forall t \in (0, T).$$

Hence

$$\sup_h \frac{1}{tp_0(\varphi, t)} q(u, t) \leq \varepsilon + \sup_{h \in X^0} \frac{1 - e^{-2K(\varepsilon)t}}{tp_0(\varphi, t)} q(u, t). \quad \forall t \in (0, T).$$

By (8.9),

$$q(u, t) \leq \varepsilon e^{2K(\varepsilon)t} tp_0(\varphi, t) \quad \forall t \in (0, T).$$

Hence

$$\sup_h \frac{1 - e^{-2K(\varepsilon)t}}{tp_0(\varphi, t)} q(u, t) \rightarrow 0 \quad \text{as } t \rightarrow 0+ \quad \forall \varepsilon > 0.$$

If  $\varphi \in X_c^0$ , then  $p_0(\varphi, t) \rightarrow \|\varphi(\cdot, 0)\|_{H^0}^2$  as  $t \rightarrow 0+$ . It follows that

$$\overline{\lim}_{t \rightarrow 0+} \sup_{\varphi \in X_c^0} \frac{q(u, t)}{t \|\varphi(\cdot, 0)\|_{H^0}} \leq \varepsilon$$

for any  $\varepsilon > 0$ . Then (5.2) follows. This completes the proof of Theorem 5.1.  $\square$

*Proof of Theorem 5.2.* Let  $\varepsilon > 0$  be given. By Corollary 3.1 again, there exists  $K(\varepsilon) = K(\varepsilon, \mathcal{P}(\mu))$  such that

$$e^{-2K(\varepsilon)t} \|u(\cdot, t)\|_{H^0}^2 \leq \left( \frac{1}{2} + \varepsilon \right) \int_0^t e^{-2K(\varepsilon)s} (F(\cdot, s), b(\cdot)^{-1} F(\cdot, s))_{H^0} ds \quad \forall t \in (0, T), \quad (8.10)$$

where  $F = (F_1, \dots, F_n)$ , and where  $F_i \in X^0$  are such that (3.1) holds. Let  $\varphi \in X_c^{-1}$ . Set

$$p(F, t) \triangleq \frac{1}{t} \int_0^t (F(\cdot, s), b(\cdot, s)^{-1} F(\cdot, s))_{H^0} ds, \quad q(u, t) \triangleq \|u(\cdot, t)\|_{H^0}.$$

It follows that

$$\sup_h \left( \frac{q(u, t)}{tp(F, t)} - \frac{1 - e^{-2K(\varepsilon)t}}{tp(F, t)} q(u, t) \right) \leq \left( \frac{1}{2} + \varepsilon \right) \quad \forall t \in (0, T).$$

Hence

$$\sup_{F: \varphi \in X_c^{-1}} \frac{1}{tp(F, t)} q(u, t) \leq \left( \frac{1}{2} + \varepsilon \right) + \sup_{h \in X^0} \frac{1 - e^{-2K(\varepsilon)t}}{tp(F, t)} q(u, t). \quad \forall t \in (0, T).$$

By (8.10),

$$q(u, t) \leq e^{2K(\varepsilon)t} \left( \frac{1}{2} + \varepsilon \right) tp(F, t) \quad \forall t \in (0, T).$$

Hence

$$\sup_{F: \varphi \in X_c^{-1}} \frac{1 - e^{-2K(\varepsilon)t}}{tp(F, t)} q(u, t) \rightarrow 0 \quad \text{as } t \rightarrow 0+ \quad \forall \varepsilon > 0.$$

If  $\varphi \in X_c^{-1}$ , then  $p(F, t) \rightarrow (F(\cdot, 0), b(\cdot, 0)^{-1} F(\cdot, 0))_{H^0}$  as  $t \rightarrow 0+$ . It follows that

$$\overline{\lim}_{t \rightarrow 0+} \sup_{F: \varphi \in X_c^{-1}} \frac{q(u, t)}{t (F(\cdot, 0), b(\cdot, 0)^{-1} F(\cdot, 0))_{H^0}} \leq \left( \frac{1}{2} + \varepsilon \right)$$

for any  $\varepsilon > 0$ . Then (5.4) follows. This completes the proof of Theorem 5.2.

*Proof of Theorem 6.1.* Note that  $u \in Y^1$  is the solution of the problem (6.1) if and only if  $u_K(x, t) \triangleq e^{-Kt}u(x, t)$  is the solution of the problem

$$\begin{aligned} \frac{\partial u_K}{\partial t} &= \mathcal{A}u_K - Ku_K + B_K(u_K) + \varphi_K, \quad t \in (0, T), \\ u_K(x, 0) &= 0, \quad u_K(x, t)|_{x \in \partial D} = 0, \end{aligned} \quad (8.11)$$

where  $\varphi_K(x, t) \triangleq e^{-Kt}\varphi(x, t)$ . In addition,

$$\|u\|_{Y^1} \leq e^{KT}\|u_K\|_{Y^1}, \quad \|\varphi_K\|_{X^{-1}} \leq \|\varphi\|_{X^{-1}}.$$

Therefore, the solvability and uniqueness in  $Y^1$  of problem (6.1) follows from existence of  $K > 0$  such that problem (8.11) has an unique solution in  $Y^1$ . Let us show that this  $K$  can be found.

We introduce operators  $F_K : X^{-1} \rightarrow Y^1$  such that  $u = F_K\varphi$  is the solution of problem (8.2).

Let  $g \in X^{-1}$  be such that

$$g = \varphi + B_K(w), \quad \text{where } w = F_K g. \quad (8.12)$$

In that case,  $u_K \triangleq F_K g \in Y^1$  is the solution of (8.11).

Equation (8.12) can be rewritten as  $g = \varphi + R_K(g)$ , or

$$g - R_K(g) = \varphi, \quad (8.13)$$

where the mapping  $R_K : X^{-1} \rightarrow X^{-1}$  is defined as

$$R_K(g) = B_K(F_K g).$$

Let  $w = F_K h$ , where  $h \in X^{-1}$ . By Theorem 3.1 reformulated as Lemma 8.1, for any  $\varepsilon > 0$ ,  $M > 0$ , there exists  $K(\varepsilon, M, \mathcal{P}(\mu)) \geq 0$  and a constant  $C_0 = C_0(\mathcal{P}(\mu))$  such that

$$\sup_{t \in [0, T]} \|w(\cdot, t)\|_{H^0}^2 + M \int_0^t \|w(\cdot, s)\|_{H^0}^2 ds \leq C_0 \|h\|_{X^{-1}} \quad \forall h \in X^{-1}. \quad (8.14)$$

Hence

$$\|F_K h\|_{X^0} \leq M^{-1} C_0 \|h\|_{X^{-1}}.$$

Take  $M$  and  $K$  such that  $\delta_* \triangleq C_* M^{-1} C_0 < 1$ . By (6.3), it follows that

$$\|R_K(g_1) - R_K(g_2)\|_{X^{-1}} \leq C_0 \|F_K g_1 - F_K g_2\|_{X^0} \leq \delta_* \|g_1 - g_2\|_{X^{-1}}. \quad (8.15)$$

By The Contraction Mapping Theorem, it follows that the equation (8.13) has an unique solution  $g \in X^{-1}$ . Hence problem (8.11) has an unique solution  $u_K = F_K g \in Y^1$ . This completes the proof of Theorem 6.1.  $\square$

*Proof of Theorem 6.2.* Let  $w = F_K h$ , where  $h \in X^0$ , and where  $F_k$  is the operator defined in the proof of Theorem 6.1. By Lemma 7.1 from Dokuchaev (2008), for any  $\varepsilon > 0$ ,  $M > 0$ , there exists  $K(\varepsilon, M, \mathcal{P}(\mu)) \geq 0$  and a constant  $C_0 = C_0(\mathcal{P}(\mu))$  such that

$$\sup_{t \in [0, T]} \|w(\cdot, t)\|_{H^1}^2 + M \int_0^t \|w(\cdot, s)\|_{H^1}^2 ds \leq C_0 \|h\|_{X^0} \quad \forall h \in X^0. \quad (8.16)$$

The rest of the proof of Theorem 6.2 repeats the proof of Theorem 6.1 with the replacement of  $Y^1$  for  $Y^2$ , and  $X^{-1}$  for  $X^0$ , and with  $R_K$  being a mapping  $R_K : X^0 \rightarrow X^0$ .  $\square$

*Proof of Theorem 6.3.* The proof for (i)-(iv) represents simplified versions of the proof for (v)-(vi) given below and will be omitted.

Let  $Q_t \triangleq \{(y, s) \in Q : s \leq t\}$ . Let us prove (v). We have that

$$\begin{aligned} & |B_K(u_1)(x, t) - B_K(u_2)(x, t)| \\ & \leq e^{-Kt} \int_{Q_t} |\beta(e^{Ks} u_1(y, s), x, t, y, s) - \beta(e^{Ks} u_2(y, s), x, t, y, s)| dy ds \\ & \leq C_L \int_Q |u_1(y, s) - u_2(y, s)| dy ds \leq C_L \ell_{n+1}(Q)^{1/2} \|u_1(\cdot) - u_2(\cdot)\|_{X^0} \end{aligned}$$

for all  $u_1(\cdot), u_2(\cdot) \in X^0$ . Since the domain  $Q$  is bounded, we have that

$$\|B(u_1) - B(u_2)\|_{X^{-1}} \leq \|B(u_1) - B(u_2)\|_{X^0} \leq \ell_{n+1}(Q)^{1/2} \|B(u_1) - B(u_2)\|_{L^\infty(Q)}.$$

Hence (6.3) holds.

Further, it follows from the assumptions that  $B(0) \in X^0$ . Hence  $B(u) \in X^0$  for all  $u \in X^0$ . This completes the proof of statement (v).

Let us prove (vi). By the definition,  $B(u) = \nabla \cdot \widehat{B}(u)$ , where  $\widehat{B} : X^0 \rightarrow X^0$  is a mapping similar to the one from statement (v). Then the proof is similar to the proof of statement (v).

Let us prove statement (vii). Let us assume that  $\widehat{\beta} = 0$ . We have that

$$\begin{aligned} & \|B_K(u_1) - B_K(u_2)\|_{X^{-1}}^2 \\ & \leq \int_0^T e^{-2Kt} \|\beta(e^{K\tau(t)} u_1(\cdot, \tau(t)), \cdot, \tau(t)) - \beta(e^{K\tau(t)} u_2(\cdot, \tau(t)), \cdot, \tau(t))\|_{H^0}^2 dt \\ & + \int_0^T e^{-2Kt} \|\widehat{\beta}(e^{K\tau(t)} u_1(\cdot, \tau(t)), \cdot, \tau(t)) - \widehat{\beta}(e^{K\tau(t)} u_2(\cdot, \tau(t)), \cdot, \tau(t))\|_{H^0}^2 dt \\ & \leq 2C_L^2 \int_0^T \|u_1(\cdot, \tau(t)) - u_2(\cdot, \tau(t))\|_{H^0}^2 dt \end{aligned}$$

$$\begin{aligned}
&= 2C_L^2 \int_0^T \|u_1(\cdot, \tau(t)) - u_2(\cdot, \tau(t))\|_{H^0}^2 \left( \frac{d\tau(t)}{dt} \right)^{-1} d\tau(t) \\
&\leq \delta_* 2C_L^2 \int_{\tau(\theta)}^{\tau(T)} \|u_1(\cdot, s) - u_2(\cdot, s)\|_{H^0}^2 ds \leq \delta_* 2C_L^2 \|u_1 - u_2\|_{X^0}^2.
\end{aligned}$$

By the assumptions, it follows that  $B(0) \in X^{-1}$ . Hence  $B(u) \in X^{-1}$  for all  $u \in X^0$ .

Let us prove statement (viii). We have that  $B_K(u) = B(u)$ , i.e., it is independent from  $K$ . Further,

$$\begin{aligned}
B(u) &= \widehat{B}(u) + \tilde{B}(u), \\
(\widehat{B}(u))(x, t) &= \int_{\mathbf{R}^n} \mathbb{I}_{\{x+c(x,y,t) \in D\}} u(x + c(x, y, t), t) \rho(y, t) dy, \\
(\tilde{B}(u))(x, t) &= -u(x, t) \int_{\mathbf{R}^n} \mathbb{I}_{\{x+c(x,y,t) \in D\}} \rho(y, t) dy \\
&\quad - \left( \int_{\mathbf{R}^n} \mathbb{I}_{\{x+c(x,y,t) \in D\}} c(x, y, t) \rho(y, t) dy \right)^\top \nabla u(x, t).
\end{aligned}$$

It follows from the assumptions that  $\tilde{B} : X^0 \rightarrow X^{-1}$  is a linear and continuous operator. Hence it suffices to prove that (6.3) holds for the operator  $\widehat{B}$ . We have that

$$(\widehat{B}(u))(x, t) = \int_D u(z, t) r(x, z, t) dz.$$

Clearly,  $\widehat{B}(0) = 0$ . Further, we have that

$$\begin{aligned}
\|\widehat{B}(u_1) - \widehat{B}(u_2)\|_{X^{-1}}^2 &\leq \|\widehat{B}(u_1) - \widehat{B}(u_2)\|_{X^0}^2 = \int_Q \left( \int_D (u_1(z, t) - u_2(z, t)) r(x, z, t) dz \right)^2 dx dt \\
&\leq \int_Q \left( \int_D |u_1(z, t) - u_2(z, t)|^2 dz \right) \left( \int_D |r(x, z, t)|^2 dz \right) dx dt \\
&\leq \int_0^T dt \left( \int_D |u_1(z, t) - u_2(z, t)|^2 dz \right) \int_D dx \int_D |r(x, z, t)|^2 dz \\
&\leq \left( \operatorname{ess\,sup}_{t \in [0, T]} \int_{D \times D} |r(x, z, t)|^2 dx dz \right) \|u_1 - u_2\|_{X^0}^2.
\end{aligned}$$

This completes the proof of statement (viii) and the proof of Theorem 6.3.  $\square$

*Proof of Theorem 7.1.* Repeat that  $u(x, t) = e^{Kt} u_K(x, t)$ , where  $u$  is the solution of problem (2.1) and  $u_K$  is the solution of (8.2) for  $h_K(x, t) = e^{-Kt} h(x, t)$ . Therefore, it suffices to find  $n, D, b, f, \lambda$ , such that

$$\begin{aligned}
&\forall T > 0, c > 0, K > 0 \quad \exists \varphi \in X^{-1} : \\
&\|u(\cdot, T)\|_{H^0}^2 \geq \left( \frac{1}{2} - c \right) \int_0^T (F(\cdot, t), b(\cdot, t)^{-1} F(\cdot, t))_{H^0} dt,
\end{aligned} \tag{8.17}$$



where  $u$  is the solution of problem (8.2) and  $F_i \in X^0$  are such as presented in (3.1),  $F = (F_1, \dots, F_n)$ .

Let us show that (8.17) holds for

$$n = 1, \quad D = (-\pi, \pi), \quad b(x, t) \equiv 1, \quad f(x, t) \equiv 0, \quad \lambda(x, t) \equiv 0. \quad (8.18)$$

In this case, (8.2) has the form

$$u'_t = u''_{xx} - Ku + h, \quad u(x, 0) \equiv 0, \quad u(x, t)|_{x \in \partial D} = 0, \quad (8.19)$$

Let

$$\gamma = m^2 + K, \quad \varphi_m(x, t) \triangleq m \sin(mx) e^{\gamma t}, \quad F_m(x, t) \triangleq -\cos(mx) e^{\gamma t}, \quad (8.20)$$

where  $m = 1, 2, 3, \dots$ . It can be verified immediately that the solution of the boundary value problem is

$$u(x, t) = m \sin(mx) \int_0^t e^{-\gamma(t-s)+\gamma s} ds = m \sin(mx) e^{-\gamma t} \int_0^t e^{2\gamma s} ds = m \sin(mx) e^{-\gamma t} \frac{e^{2\gamma t} - 1}{2\gamma}.$$

Hence

$$\|u(\cdot, T)\|_{H^0}^2 = m^2 \|\sin(mx)\|_{H^0}^2 e^{-2\gamma T} \left( \frac{e^{2\gamma T} - 1}{2\gamma} \right)^2 = m^2 \pi e^{-2\gamma T} \frac{(e^{2\gamma T} - 1)^2}{4\gamma^2},$$

and

$$\int_0^T \|F_m(\cdot, t)\|_{H^0}^2 dt = \|\cos(mx)\|_{H^0}^2 \int_0^T e^{2\gamma t} dt = \pi \frac{e^{2\gamma T} - 1}{2\gamma}.$$

It follows that

$$\|u(\cdot, T)\|_{H^0}^2 \left( \int_0^T \|F_m(\cdot, t)\|_{H^0}^2 dt \right)^{-1} = \frac{m^2}{2\gamma} e^{-2\gamma T} (e^{2\gamma T} - 1) = \frac{m^2}{2\gamma} (1 - e^{-2\gamma T}) \rightarrow \frac{1}{2} \quad (8.21)$$

as  $\gamma \rightarrow +\infty$ . In particular, it holds if  $K$  is fixed and  $m \rightarrow +\infty$ . It follows that (7.1) holds. This completes the proof of Theorem 7.1.  $\square$

*Proof of Theorem 7.2.* Let the parameters be defined by (8.18). Consider a sequence  $\{T_i\}$  such that  $T_i \rightarrow 0+$  as  $i \rightarrow +\infty$ . Let  $\varphi = \varphi_m$  be defined by (8.20) for an increasing sequence of integers  $m = m_i$  such that  $m_i > T_i^{-1}$ . In that case,  $\gamma T \rightarrow +\infty$ . Hence (8.21) holds and (7.2) holds.  $\square$

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